FOURIER-ovi redovi - matematički dio

Složeno periodična funkcija $f(t) = f(t+T)$ može se predstaviti Fourier-ovim redom, koji predstavlja sumu prostoperiodičnih funkcija u obliku:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{15.702}$$

gdje je $\omega_0 = 2\pi/T$ osnovna ili fundamentalna učestanost. Oblik dat relacijom (15.702) naziva se **trigonometrijski** oblik Fourier-ovog reda. Pojedini članovi Fourier-ovog reda nazivaju se harmonicima. Član učestanosti ω_0 naziva se osnovnim harmonikom, a član učestanosti $n\omega_0$ n - tim harmonikom. Član $\frac{a_0}{2}$ naziva se nultim harmonikom. a_0, a_n i b_n - nazivaju se koeficijenti Fourier-ovog reda i oni se izračunavaju po formulama:

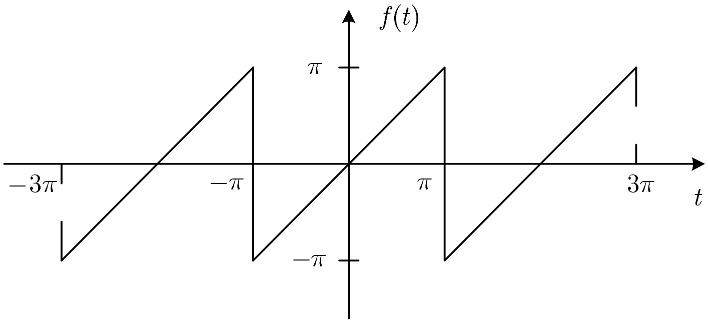
$$\begin{aligned}
a_0 &= \frac{2}{T}\int\limits_{t_0}^{t_0+T} f(t)dt \\
a_n &= \frac{2}{T}\int\limits_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \quad n = 0, 1, 2, ... \\
b_n &= \frac{2}{T}\int\limits_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \quad n = 1, 2, ...
\end{aligned}$$

Da bi se neka složenoperiodična funkcija razvila u Fourier-ov red ona treba da ispuni Dirchlet-ove uslove:

1. Da ima prekide prve vrste i to konačan broj.
2. Da ima konačan broj minimuma i maksimuma.
3. Da je $\int\limits_0^t |f(t)|\, dt < 0$

U tačkama prekida red konvergira vrijednosti $\frac{1}{2}[f(t^+) + f(t^-)]$.

Primjer 1: Razviti u Fourier-ov red funkciju $f(t) = t$; $-\pi < t < \pi$; $f(t + 2\pi) = f(t)$ što znači da je $T = 2\pi$ (slika 15.323).



Slika 15.323:

Koeficijenti su:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t dt = 0$$

Za $n = 1, 2, 3, ...$ imamo da je:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \cos nt dt = \frac{1}{n^2\pi}\left(\cos nt + nt \sin nt\right)\Big|_{-\pi}^{\pi} = 0$$

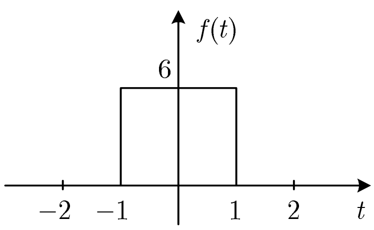
i

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt dt = \frac{1}{n^2\pi}\left(\sin nt - nt \cos nt\right)\Big|_{-\pi}^{\pi} = -\frac{2\cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$$

Prema tome, funkcija $f(t)$ zapisana preko Fourier-ovog reda je:

$$f(t) = 2\left(\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - ...\right)$$

Primjer 2: Razviti u Fourier-ov red funkciju $f(t)$ koja je jednaka (slika 15.324.):



Slika 15.324:

$$f(t)=\begin{cases} 0 & -2<t<-1 \\ 6 & -1<t<1 \\ 0 & 1<t<2 \end{cases}$$

$f(t+4)=f(t)\Rightarrow T=4\,;\ \omega_0=\frac{2\pi}{T}=\frac{\pi}{2}$. Koeficijenti su:

$$a_0=\frac{2}{4}\int_{-1}^{1}6dt+\frac{2}{4}\int_{1}^{3}0dt=6$$

$$a_n=\frac{2}{4}\int_{-1}^{1}6\cos\frac{n\pi t}{2}dt+\frac{2}{4}\int_{1}^{3}0\cos\frac{n\pi t}{2}dt=\frac{12}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n=\frac{2}{4}\int_{-1}^{1}6\sin\frac{n\pi t}{2}dt+\frac{2}{4}\int_{1}^{3}0\sin\frac{n\pi t}{2}dt=0$$

$$f(t)=3+\frac{12}{\pi}\left(\cos\frac{\pi t}{2}-\frac{1}{3}\cos\frac{3\pi t}{2}+\frac{1}{5}\cos\frac{5\pi t}{2}-...\right)$$

$$f(t)=3+\frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\{[(2n-1)\,\pi t]\,/2\}}{2n-1}$$

Primjer 3: Razviti u Fourier-ov red funkciju $f(t)$ koja je jednaka:

$$f(t)=\begin{cases} 4 & 0<t<1 \\ -4 & 1<t<2 \end{cases}$$

$f(t)=f(t+2);\ T=2$

Koeficijenti su jednaki:

$$a_n=0;\quad n=0,1,2,...$$

$$b_n=\frac{4}{2}\int_{0}^{1}4\sin n\pi t dt=\frac{8}{n\pi}\left[1-(-1)^n\right]\Rightarrow\begin{cases} b_n=0; & n\text{ - parno} \\ b_n=\frac{16}{n\pi}; & n\text{ - neparno} \end{cases}$$

$$f(t)=\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi t}{2n-1}$$

Oblik Fourier-ovog reda zavisi od svojstva vremenske funkcije $f(t)$:

1. Ako je funkcija $f(t)$ parna tj. $f(-t)=f(t)$ Fourier-ov red će sadržati konstantni član i

kosinusne članove:

$$\begin{aligned} a_n &= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_0 t dt \quad n = 0,1,2,... \\ b_n &= 0 \qquad\qquad\qquad\qquad\qquad n = 1,2,3,... \end{aligned}$$

2. Ako je funkcija $f(t)$ neparna tj. $f(-t) = -f(t)$ Fourier-ov red će sadržati samo sinusne članove:

$$\begin{aligned} a_n &= 0 \qquad\qquad\qquad\qquad\qquad n = 1,2,3,... \\ b_n &= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\omega_0 t dt \quad n = 0,1,2,... \end{aligned}$$

3. Ako je negativni talas vremenske funkcije ogledalska slika pozitivnog talasa tj. ako je: $f(t + T/2) = -f(t)$ onda je:

$$\begin{aligned} a_n &= 0 \qquad\qquad\qquad\qquad\qquad \text{za } n \text{ - parno} \\ a_n &= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_0 t dt \quad \text{za } n \text{ - neparno} \\ b_n &= 0 \qquad\qquad\qquad\qquad\qquad \text{za } n \text{ - parno} \\ b_n &= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\omega_0 t dt \quad \text{za } n \text{ - neparno} \end{aligned}$$

tj. Fourier-ov red će sadržavati samo neparne članove.

4. Ako funkcija $f(t)$ ispunjava uslov iz trećeg slučaja i uz to je simetrična u odnosu na koordinatni početak, tj. neparna, njen Fourier-ov red imaće samo neparne sinusne harmonike:

$$b_{2n+1} = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\sin(2n+1)\omega_0 t dt \qquad n = 1,2,3,...$$

Primjer 4: Razviti u Fourier-ov red funkciju $f(t)$:

$$f(t) = \begin{cases} 0 & -\frac{\pi}{2} < t < -\frac{\pi}{4} \\ 4\cos 2t & -\frac{\pi}{4} < t < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < t < \frac{\pi}{2} \end{cases}$$

$$f(t+\pi) = f(t)$$

$$
\begin{aligned}
a_0 &= \frac{8}{\pi}; \quad a_1 = 2; \quad a_2 = \frac{8}{\pi(1-n^2)} \cos\frac{n\pi}{2}, \quad n = 2, 3, 4, ... \\
b_n &= 0, \quad n = 1, 2, 3, ...
\end{aligned}
$$

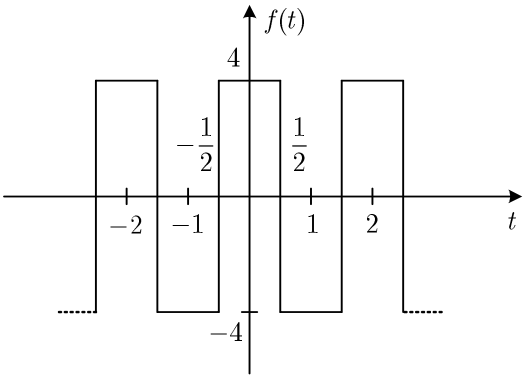
Primjer 5: Razviti u Fourier-ov red funkciju $f(t)$:

$$f(t) = |4 \sin 2t|$$

Odgovor:

$$f(t) = \frac{16}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos 4nt\right)$$

Primjer 6: Razviti u Fourier-ov red funkciju $f(t)$ prikazanu na slici 15.325.



Slika 15.325:

Odgovor:

$$f(t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos(2n-1)\pi t$$

Drugi oblici trigonometrijskog Fourier-ovog reda su:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(n\omega_0 t + \theta_n\right) \tag{15.703}$$

Koristeći sledeći identitet:

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \cos\left(n\omega_0 t + \theta_n\right)$$

dobijamo izraze za A_n i θ_n koji su jednaki:

$$\begin{aligned} A_n &= \sqrt{a_n^2 + b_n^2} \\ \theta_n &= \arctan \frac{b_n}{a_n} \end{aligned}$$

iz kojih slijedi:

$$\begin{aligned} a_n &= A_n \cos \theta_n \\ b_n &= -A_n \sin \theta_n \end{aligned}$$

ili:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(n\omega_0 t + \theta_n^{'}\right) \tag{15.704}$$

gdje je:

$$\theta_n^{'} = \theta_n + \frac{\pi}{2}$$

Kompleksni oblik Fourier-ovog reda

Polazeći od relacija:

$$\begin{aligned} \cos n\omega_0 t &= \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) \\ \sin n\omega_0 t &= \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) \end{aligned}$$

i oblika (15.702) Fourier-ovog reda dobijamo:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega_0 t} \right]$$

Uvodeći novi koeficijent $\underline{c}_n$ koji je jednak:

$$\underline{c}_n = \frac{a_n - jb_n}{2}$$

Zamjenjujući izraze za izračunavanje konstanti a_n i b_n ako je $t_0 = -T/2$ dobijamo:

$$\underline{c}_n = \frac{1}{T} \int\limits_{-\frac{T}{2}}^{\frac{T}{4}} f(t) \left(\cos n\omega_0 t - j \sin n\omega_0 t\right) dt$$

ili koristeći Euler-ovu formulu:

$$\underline{c}_n = \frac{1}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{4}} f(t)e^{-jn\omega_0 t}dt \tag{15.705}$$

$$\underline{c}_n^* = \frac{a_n + jb_n}{2} = \frac{1}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{4}} f(t)\left(\cos n\omega_0 t + j\sin n\omega_0 t\right)dt$$

Evidentno je $\underline{c}_{-n}$ (ako n zamijenimo sa $-n$) da je:

$$\underline{c}_{-n} = \frac{a_n + jb_n}{2} = \underline{c}_n^*$$

$$\frac{a_0}{2} = \frac{1}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{4}} f(t)dt = c_0$$

Sada možemo napisati:

$$f(t) = c_0 + \sum_{n=1}^{\infty}\underline{c}_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\underline{c}_{-n} e^{jn\omega_0 t} \tag{15.706}$$

Pošto je $c_0 = \underline{c}_n|_{n=0}$ relaciju (15.706) možemo zapisati u sledećem obliku:

$$f(t) = \sum_{n=0}^{\infty}\underline{c}_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty}\underline{c}_n e^{jn\omega_0 t}$$

Pa konačno dobijamo:

$$f(t) = \sum_{n=-\infty}^{\infty}\underline{c}_n e^{jn\omega_0 t} \tag{15.707}$$

Relacija (15.707) predstavlja kompleksni oblik Fourier-ovog reda dok se koeficijent $\underline{c}_n$ izračunava koristeći relaciju (15.705).

Primjer 7: Naći kompleksni oblik Fourier-ovog reda funkcije $f(t)$:

$$f(t) = \begin{cases} 4 & 0 < t < 1 \\ -4 & 1 < t < 2 \end{cases}$$

ako je $f(t+2) = f(t)$. Odavde slijedi da je: $T = 2$ i $\omega_0 = 2\pi/T = \pi$ pa je:.

$$\underline{c}_n = \frac{1}{2}\int\limits_{-1}^{1} f(t)e^{-jn\pi t}dt = \frac{1}{2}\int\limits_{-1}^{0}(-4)e^{-jn\pi t}dt + \frac{1}{2}\int\limits_{0}^{1} 4e^{-jn\pi t}dt = \frac{4}{jn\pi}\left[1 - (-1)^n\right]$$

Takođe, imamo da je:

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(t)dt = \frac{1}{2}\int_{-1}^{0} 4dt - \frac{1}{2}\int_{0}^{1} 4dt = 0$$

Pošto je $\underline{c}_n = 0$ za parno n i $\underline{c}_n = 8/jn\pi$ za neparno n kompleksni oblik Fourier-ovog reda funkcije $f(t)$ ima sledeći oblik:

$$f(t) = \frac{8}{jn\pi}\sum_{n=-\infty}^{\infty} \frac{1}{2n-1} e^{j(2n-1)\pi t}$$

Primjer 8: Naći kompleksni oblik Fourier-ovog reda funkcije $f(t)$:

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

ako je $f(t+4) = f(t)$:

Odgovor:

$$f(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n} e^{j\frac{n\pi t}{2}}$$

Funkcija

$$\begin{aligned} \mathrm{Sa}(x) &= \frac{\sin x}{x}, \quad x \neq 0 \\ \mathrm{Sa}(0) &= 1 \end{aligned}$$

naziva se **funkcija odabiranja** (sampling funkcija). Funkcija:

$$\mathrm{sinc}x = \frac{\sin \pi x}{\pi x} = \mathrm{Sa}(\pi x)$$

naziva se **sinc** funkcija. Pošto je:

$$c_0 = \lim_{n\to 0} \underline{c}_n$$

rezultat iz **Primjera 8** se može napisati u sledećem obliku:

$$f(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{Sa}\left(\frac{n\pi}{2}\right) e^{j\frac{n\pi t}{2}}$$

Primjer 9:Naći kompleksni oblik Fourier-ovog reda funkcije $f(t)$:

$$f(t) = \begin{cases} 1 & -\frac{b}{2} < t < \frac{b}{2} \\ 0 & \frac{b}{2} < t < \frac{T}{2} \end{cases}$$

ako je $f(t+T) = f(t)$:

Odgovor:

$$f(t) = \frac{b}{T} \sum_{n=-\infty}^{\infty} \mathrm{Sa}\left(\frac{n\pi b}{T}\right) e^{j\frac{n\pi t}{T}}$$

Ako je funkcija $f(t)$ parna, koeficijenti kompleksnog oblika Fourier-ovog reda su jednaki:

$$\underline{c}_n = \frac{a_n}{2} = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_0 t dt$$

Ako je funkcija $f(t)$ neparna, koeficijenti kompleksnog oblika Fourier-ovog reda su jednaki:

$$\underline{c}_n = \frac{b_n}{2j} = \frac{2}{jT}\int_0^{\frac{T}{2}} f(t)\sin n\omega_0 t dt$$

Ako je funkcija $f(t)$ polu-talasno simetrična (half-wave simetry) tada su koeficijenti kompleksnog oblika Fourier-ovog reda su jednaki:

$$\begin{aligned} \underline{c}_n &= 0 & n \text{ - parno} \\ \underline{c}_n &= \frac{2}{T}\int_0^{\frac{T}{2}} f(t)e^{-jn\omega_0 t} & n \text{ - neparno} \end{aligned}$$

Frekventni spektar

Razmatramo kompleksni (eksponencijalni) oblik Fourier-ovog reda:

$$f(t) = \sum_{n=-\infty}^{\infty} \underline{c}_n e^{jn\omega_0 t}$$

gdje je:

$$\underline{c}_n = \frac{a_n - jb_n}{2}$$

ili u polarnim koordinatama:

$$\underline{c}_n = \frac{\sqrt{a_n^2 + b_n^2}}{2}\angle - \tan^{-1}\frac{b_n}{a_n}$$

U funkciji od A_n i θ_n imamo da je:

$$\underline{c}_n = \frac{A_n}{2}\angle\theta_n$$

$$\underline{c}_{-n} = \frac{A_n}{2}\angle - \theta_n$$

Takođe, imamo da je:

$$c_o = A_n \angle \theta_n = \frac{a_0}{2}$$

Na osnovu predhodnih relacija možemo pisati da je:

$$\begin{aligned} |\underline{c}_n| &= \left|\underline{c}_{-n}\right| = \frac{A_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2}, \qquad n = 1, 2, 3, ... \\ |c_0| &= A_0 = \frac{a_0}{2} \end{aligned}$$

gdje je: $|\underline{c}_n|$ – **Diskretni amplitudski spektar** a θ_n– **Diskretni fazni spektar**.

Neke osobine Fourier-ovih redova

1. **Linearnost**

Ako su $x(t)$ i $y(t)$ dva periodična signala sa periodom T tada važi:

$$\begin{aligned} x(t) &\overset{\mathrm{F}\mathcal{S}}{\rightleftharpoons} \underline{a}_k \\ y(t) &\overset{\mathrm{F}\mathcal{S}}{\rightleftharpoons} \underline{b}_k \\ z(t) \quad = \quad Ax(t) + By(t) &\overset{\mathrm{F}\mathcal{S}}{\rightleftharpoons} \underline{c}_k = A\underline{a}_k + B\underline{b}_k \end{aligned}$$

Ova osobina je direktna posledica relacija za Fourier-ov red u kompleksnom obliku.

2. **Pomak (Time Shifting)**

$$\begin{aligned} x(t) &\overset{\mathrm{F}\mathcal{S}}{\rightleftharpoons} \underline{a}_k \\ x(t - t_0) &\overset{\mathrm{F}\mathcal{S}}{\rightleftharpoons} e^{-jk\omega_0 t_0}\underline{a}_k \end{aligned}$$

Dokaz:

$$\underline{b}_k = \frac{1}{T}\int_0^T x(t - t_0)e^{-jk\omega_0 t}dt \tag{15.708}$$

Uvodeći smjenu $\tau = t - t_0$ u relaciju (15.708) dobijamo:

$$\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0(\tau + t_0)}d\tau = e^{-jk\omega_0 t_0}\underbrace{\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0 \tau}d\tau}_{\underline{a}_k} = e^{-jk\omega_0 t_0}\underline{a}_k$$

$$|\underline{b}_k| = |\underline{a}_k|$$

3. **Time reversal**

$$y(t) = x(-t)$$

$$x(-t) = \sum_{k=-\infty}^{\infty} \underline{a}_k e^{-jk\omega_0 t}$$

Ako uvedemo smjenu $k = -m$ dobijamo:

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} \underline{a}_{-m} e^{jm\omega_0 t}$$

$$\underline{b}_k = \underline{a}_{-k}$$

$$x(t) \overset{\mathrm{F}\mathcal{S}}{\risingdotseq} \underline{a}_k$$

$$x(-t) \overset{\mathrm{F}\mathcal{S}}{\risingdotseq} \underline{a}_{-k}$$

Posledice: Ako je funkcija $x(t)$ parna tj: $x(-t) = x(t)$ onda je $\underline{a}_{-k} = \underline{a}_k$. Ako je funkcija $x(t)$ neparna tj: $x(-t) = -x(t)$ onda je $\underline{a}_{-k} = -\underline{a}_k$.

4. **Conjugation and conjugate symmetry**

Ako je:

$$x(t) \overset{\mathrm{F}\mathcal{S}}{\risingdotseq} \underline{a}_k$$

tada je

$$x^*(t) \overset{\mathrm{F}\mathcal{S}}{\risingdotseq} \underline{a}^*_{-k}$$

Dokaz:

$$x(t) = \sum_{k=-\infty}^{\infty} \underline{a}_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} \underline{a}^*_k e^{-jk\omega_0 t}$$

Zamjenjujući $k \to -k$ dobijamo:

$$x^*(t) = \sum_{k=-\infty}^{\infty} \underline{a}^*_{-k} e^{jk\omega_0 t}$$

Posledice: Ako je $x(t)$ realan signal onda je $x(t) = x^*(t)$ pa imamo:

$$\underline{a}_{-k} = \underline{a}_k^* \tag{15.709}$$

Relacija (15.709) izražava **conjugate symmetry.**

$$|\underline{a}_k| = |\underline{a}_{-k}|$$

Ako je funkcija $x(t)$ realna i parna imamo

$$\underline{a}_k = \underline{a}_{-k}; \quad \underline{a}_k^* = \underline{a}_{-k} \Rightarrow \underline{a}_k = \underline{a}_k^*$$

5. **Time scaling**

Ako je $x(t)$ periodična funkcija sa osnovnim periodom T i učestanošću $\omega_0 = 2\pi/T$ i $x(\alpha t)$ sa fundamentalnim periodom T/α i učestanošću $\omega = \alpha\ \omega_0$ gdje je α realno i pozitivno imano da je:

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} \underline{a}_k e^{jk(\alpha\omega_0)t}$$

i koeficijenti Fourier-ovog reda se ne **mijenjaju**.

6. **Teorema konvolucije**

Ako imamo dvije funkcije $f_1(t) = \sum\limits_{n=-\infty}^{\infty} \underline{a}_n e^{jn\omega_0 t}$ i $f_2(t) = \sum\limits_{n=-\infty}^{\infty} \underline{b}_n e^{jn\omega_0 t}$ sa istim periodom T i učestanošću $\omega_0 = 2\pi/T$ tada je:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(\tau) f_2(t-\tau) d\tau = \sum_{n=-\infty}^{\infty} \underline{a}_n \underline{b}_n e^{jn\omega_0 t} \tag{15.710}$$

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(t) f_2(t) e^{-jn\omega_0 t} dt = \sum_{m=-\infty}^{\infty} \underline{a}_m \underline{b}_{n-m} \tag{15.711}$$

Dokaz:

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(t-\tau) f_2(\tau) d\tau &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} \sum_{n=-\infty}^{\infty} \underline{a}_n e^{jn\omega_0(t-\tau)} f_2(\tau) d\tau = \sum_{n=-\infty}^{\infty} \underline{a}_n e^{jn\omega_0 t} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} f_2(\tau) e^{-jn\omega_0 \tau} d\tau = \\ &= \sum_{n=-\infty}^{\infty} \underline{a}_n \underline{b}_n e^{jn\omega_0 t}\end{aligned}$$

$$
\begin{aligned}
f_t(t)f_2(t) &= \sum_{m=-\infty}^{\infty} \underline{a}_m e^{jm\omega_0 t} \sum_{k=-\infty}^{\infty} \underline{b}_k e^{jk\omega_0 t} = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \underline{a}_m \underline{b}_k e^{j(m+k)\omega_0 t} = \\
&= \sum_{m=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \underline{a}_{m-k} \underline{b}_k \right) e^{jm\omega_0 t}
\end{aligned}
$$

7. **Parseval-ova teorema**

Ako su $f_1(t)$ i $f_2(t)$ dvije periodične funkcije sa istim periodom T onda je:

$$
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(t) f_2^*(t) dt = \sum_{m=-\infty}^{\infty} \underline{a}_m \underline{b}_m^* \tag{15.712}
$$

Dokaz: Na osnovu izvedenih predhodnih osobina koeficijenti Fourier-ovog reda funkcije $f_2^*(t)$ su $\underline{b}^*_{-m}$. Ako u relaciju (15.711) umjesto funkcije $f_2(t)$ stavimo $f_2^*(t)$ dobijamo:

$$
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(t) f_2^*(t) e^{-jn\omega_0 t} dt = \sum_{m=-\infty}^{\infty} \underline{a}_m \underline{b}_{m-n}
$$

Stavljajući $n = 0$ dobijamo:

$$
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{4}} f_1(t) f_2^*(t) = \sum_{m=-\infty}^{\infty} \underline{a}_m \underline{b}_m
$$

Posledica ove teoreme je je teorema **Releja** ili **Energy theorema**.

$$
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{4}} |f_1(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\underline{a}_n|^2 \tag{15.713}
$$

Dokaz: Ako u teoremi Parsevala stavimo $f_1(t) = f_2(t) = f(t)$ dobijamo relaciju (15.713)

Napomena: Na osnovu veza između različitih formi Fourier-ovog reda teoremu Releja je moguće napisati i na sledeći način:

$$
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{4}} f^2(t) dt = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{A_n^2}{2} = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} \tag{15.714}
$$

Relacija (15.714) se ponegdje u literaturi naziva i Parserval-ovom jednačinom.

FOURIER-ova transformacija - matematički dio

Direktna Fourier-ova transformacija neke vremenske funkcije se definiše kao:

$$F(j\omega) = \int\limits_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

To se može napisati i označiti na sledeći način:

$$\mathrm{F}\left\{f(t)\right\} = F(j\omega) = \int\limits_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

ili kraće:

$$f(t) \overset{\mathrm{F}}{\risingdotseq} F(j\omega)$$

Inverzna Fourier-ova transformacija se definiše na sledeći način:

$$f(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega$$

ili:

$$f(t) = \mathrm{F}^{-1}\left\{F(j\omega)\right\} = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega$$

Da bi funkcija $f(t)$ imala Fourier-ovu transformaciju ona treba da zadovolji Dirichlet-ove uslove i uslov apsolutne integrabilnosti: $\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty$.

Primjer 1: Odrediti Fourier-ovu transformaciju funkcije: $f(t) = e^{-at}h(t)$ gdje je $a > 0$.

$$\begin{aligned}\mathrm{F}\left\{e^{-at}h(t)\right\} &= \int\limits_{-\infty}^{+\infty} e^{-at}h(t)e^{-j\omega t}dt = \int\limits_{0}^{+\infty} e^{-(a+j\omega)t}dt = \\ &= \frac{1}{-(a+j\omega)}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}\end{aligned}$$

jer je zbog $a > 0$ $\lim_{t \to \infty} e^{-at} (\cos \omega t - j \sin \omega t) = 0$.

Primjer 2: Odrediti Fourier-ovu transformaciju funkcije:

$$f(t) = \begin{cases} A & -\frac{a}{2} < t < \frac{a}{2} \\ 0 & |t| > \frac{a}{2} \end{cases}$$

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = \int_{-\frac{a}{2}}^{\frac{a}{2}} A e^{-j\omega t} dt = \frac{2A}{\omega} \left(\frac{e^{j\frac{\omega a}{2}} - e^{-j\frac{\omega a}{2}}}{j2} \right) = \\ &= \frac{2A}{\omega} \sin \frac{\omega a}{2} = A a S_a \left(\frac{\omega a}{2} \right) \end{aligned}$$

Primjer 3: Odrediti Fourier-ovu transformaciju funkcije:

$$f(t) = \begin{cases} t+1 & -1 < t < 0 \\ -t+1 & 0 < t < 1 \\ 0 & \text{van tih intervala} \end{cases}$$

$$\mathrm{F}\{f(t)\} = F(j\omega) = \frac{2(1-\cos\omega)}{\omega^2} = S_a^2 \left(\frac{\omega}{2} \right)$$

Primjer 4: Odrediti Fourier-ovu transformaciju funkcije: $f(t) = e^{-a|t|}$ gdje je $a > 0$.

$$\mathrm{F}\{f(t)\} = F(j\omega) = \frac{2a}{\omega^2 + a^2}$$

Primjer 5: Odrediti inverznu Fourier-ovu transformaciju funkcije:

$$F(j\omega) = \begin{cases} 1 & -1 < \omega < 1 \\ 0 & |\omega| > 1 \end{cases}$$

$$f(t) = \mathrm{F}^{-1}\{F(j\omega)\} = \frac{1}{\pi} S_a(t)$$

Primjer 6: Odrediti Fourier-ovu transformaciju funkcije: $f(t) = \delta(t)$.

$$\mathrm{F}\{\delta(t)\} = F(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

Ako formiramo funkciju:

$$f_\sigma(t) = \frac{1}{2\pi} \int_{-\sigma}^{+\sigma} F(j\omega) e^{j\omega t} d\omega$$

tada je:

$$f_\sigma(t) \;\; \overrightarrow{\sigma \to \infty} \;\; f(t)$$

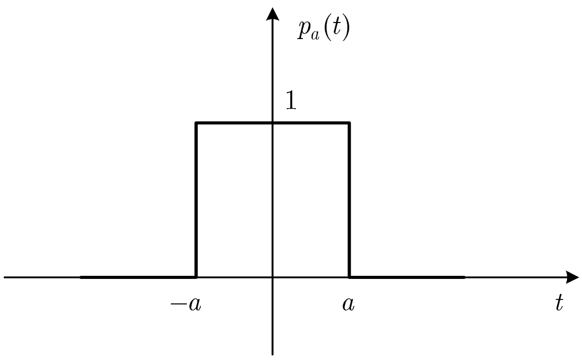
$$f_\sigma(t) = \frac{1}{2\pi}\int_{-\sigma}^{+\sigma} e^{j\omega t}\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\tau)\int_{-\sigma}^{+\sigma} e^{j\omega(t-\tau)}d\omega d\tau$$

$$f_\sigma(t) = \int_{-\infty}^{+\infty} f(\tau)\frac{\sin\sigma\,(t-\tau)}{\pi\,(t-\tau)}d\tau$$

Gornji integral jednak je konvoluciji funkcije $f(t)$ i FOURIER - INTEGRAL KERNAL $\frac{\sin\sigma t}{\pi t}$ i on teži za $\sigma \to \infty$ vremenskoj funkciji $f(t)$. Ako $f(t)$ ima diskontinuitet u t onda je:

$$f_\sigma(t) \xrightarrow[\sigma \to \infty]{} \frac{f(t^+) + f(t^-)}{2}$$

Primjer 7: Jedinični pravougaoni impuls $p_a(t)$ ima grafik prikazan na slici 15.326.



Slika 15.326: Jedinični pravougaoni impulas

koji je definisan sa:

$$p_a(t) = \begin{cases} 1 & |t| < 0 \\ 0 & |t| > 0 \end{cases}$$

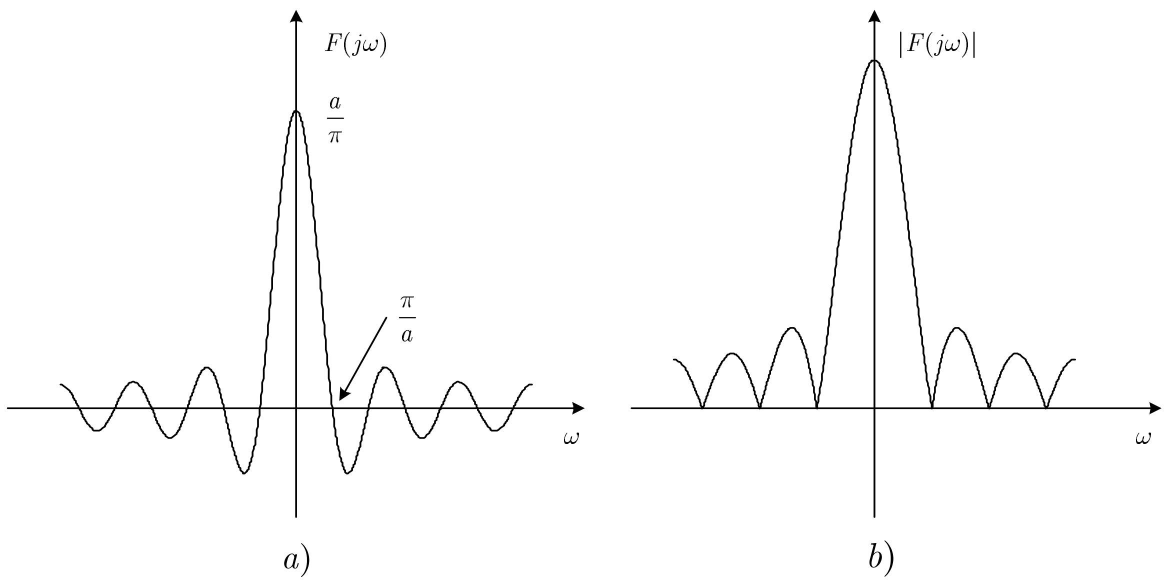
Fourier-ova transformacija jediničnog impulsa je:

$$F(j\omega) = \mathrm{F}\left\{p_a(t)\right\} = \int_{-a}^{+a} 1\; e^{-j\omega t}dt = \frac{2\sin a\omega}{\omega}$$

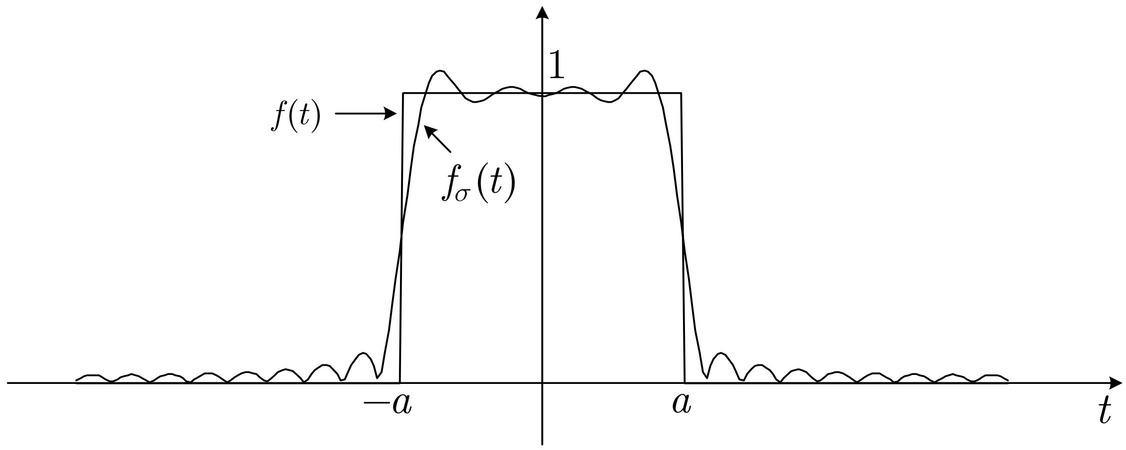
i prikazana je na slici 15.327:

$$f_\sigma(t) = \int_{-a}^{+a} 1\;\frac{\sin\sigma\,(t-\tau)}{\pi\,(t-\tau)}d\tau = \frac{1}{\pi}\left\{\mathrm{Si}\left[\sigma(t+a)\right] - \mathrm{Si}\left[\sigma(t-a)\right]\right\}$$

Sa slike 15.328. se vidi da u svakom opsegu postoje odstupanja koja imaju oscilatorni karakter. Ovaj fenomen se u literaturi naziva **Gibsove oscilacije**.



Slika 15.327: $a)$ $F(j\omega)-$ Realni dio frekevencijskog spektra; $b)$ $|F(j\omega)| -$ Amplitudski spektar



Slika 15.328: Gibsove oscilacije

$$\mathrm{Si}(t) = \int\limits_0^t \frac{\sin\tau}{\tau} d\tau$$

Funkcija $\mathrm{Si}(t)$ naziva se INTERRA SINUS.

Oblici Fourier-ove transformacije

Ako je u opštem slučaju vremenska funkcija kompleksna:

$$f(t) = f_1(t) + jf_2(t)$$

tada je

$$F\{f(t)\} = F(j\omega) = R(\omega) + jX(\omega)$$

Iz definicionih oblika direktne i inverzne Fourier-ove transformacije dobijamo:

$$R(\omega) = \int_{-\infty}^{+\infty} [f_1(t)\cos\omega t + f_2(t)\sin\omega t]\, dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} [f_2(t)\cos\omega t - f_1(t)\sin\omega t]\, dt$$

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [R(\omega)\cos\omega t - X(\omega)\sin\omega t]\, d\omega$$

$$f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [R(\omega)\sin\omega t + X(\omega)\cos\omega t]\, d\omega$$

Kada je $f(t)$ realan signal onda je: $f(t) = f_1(t)$; $f_2(t) = 0$ pa imamo drugi oblik Fourier-ovih transformacija:

$$F(j\omega) = R(\omega) + jX(\omega)$$

$$R(\omega) = \int_{-\infty}^{+\infty} f(t)\cos\omega t dt \tag{15.715}$$

$$X(\omega) = -\int_{-\infty}^{+\infty} f(t)\sin\omega t dt \tag{15.716}$$

Relacija (15.715) predstavlja **kosinusnu transformaciju** a relacija (15.716) predstavlja **sinusnu transformaciju**.

$$f(t) = f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [R(\omega)\cos\omega t - X(\omega)\sin\omega t]\, d\omega$$

Pošto je: $R(-\omega) = R(\omega)$; $X(-\omega) = -X(\omega)$, onda je za realni spektar:

$$F^*(j\omega) = F(-j\omega)$$

Ako je $f(t)$ parna funkcija onda je $X(\omega) = 0$, odnosno:

$$F(j\omega) = R(\omega) = \int_{-\infty}^{+\infty} f(t) \cos \omega t dt = 2 \int_{0}^{+\infty} f(t) \cos \omega t dt \qquad (15.717)$$

Iz relacije (15.717) slijedi da je Fourier-ova transformacija realne parne funkcije realna. Ako je $f(-t) = -f(t)$ tj. ako je realna funkcija neparna tada je $R(\omega) = 0$, odnosno:

$$F(j\omega) = jX(\omega) = -j \int_{-\infty}^{+\infty} f(t) \sin \omega t dt = -2j \int_{0}^{+\infty} f(t) \sin \omega t dt \qquad (15.718)$$

To znači da je Fourier-ova transformacija realne neparne funkcije čisto imaginarna.

Primjer 8: Funkcija $f(t) = \frac{1}{t}$ je neparna funkcija. Fourier-ova transformacija je jednaka:

$$F(j\omega) = -j \int_{-\infty}^{+\infty} \frac{\sin \omega t}{t} dt = \begin{cases} -j\pi & \omega > 0 \\ j\pi & \omega < 0 \end{cases}$$

To se može napisati i na sledeći način:

$$\mathrm{F}\left\{\frac{1}{\pi t}\right\} \fallingdotseq -j\mathrm{sgn}\omega$$

Primjer 9: Odrediti Fourier-ovu transformaciju funkcije $f(-t) =?$

$$\mathrm{F}\left\{f(-t)\right\} = \int_{-\infty}^{+\infty} f(-t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt = F(-j\omega)$$

pošto je $F(-j\omega) = R(\omega) - jX(\omega)$.

Ako je $f(t)$ realno onda važi da je:

$$f(t) \fallingdotseq R(\omega) + jX(\omega) \ \text{ i } \ f(-t) \fallingdotseq R(\omega) - jX(\omega).$$

Tada je parni dio od $f(t)$ jednak je:

$$f_e(t) = \mathrm{Ev}\left\{f(t)\right\} = \frac{f(t) + f(-t)}{2}$$

dok je neparni dio funkcije $f(t)$ jednak:

$$f_o(t) = \mathrm{Odd}\left\{f(t)\right\} = \frac{f(t) - f(-t)}{2}$$

Na osnovu predhodnog važi da je: $f_e(t) \fallingdotseq R(\omega)$ i $f_o(t) \fallingdotseq jX(\omega)$. Ako je $f(t)$ kauzalna funkcija tj. ako je $f(t) = 0$ za $t < 0$ tada je:

$$f(t) = 2f_e(t) = 2f_o(t) \quad \text{za} \quad t > 0$$

Ako je funkcija $f(t)$ kauzalna realna funkcija onda je ona jednoznačno određena preko $R(\omega)$ ili $X(\omega)$:

$$f(t) = \frac{2}{\pi}\int_0^\infty R(\omega)\cos\omega\; d\omega = -\frac{2}{\pi}\int_0^\infty X(\omega)\sin\omega\; d\omega; \qquad t > 0$$

Primjer 10:

$$e^{-\alpha t}h(t) \risingdotseq \frac{1}{\alpha + j\omega} = \frac{\alpha}{\alpha^2 + \omega^2} - j\frac{\omega}{\alpha^2 + \omega^2}$$

Primjer 11.

$$e^{-\alpha|t|} \risingdotseq \frac{2\alpha}{\alpha^2 + \omega^2}$$

zato što je $e^{-\alpha|t|}$ parni dio od funkcije $2e^{-\alpha t}h(t)$.

Polazeći od drugog oblika Fourier-ove transformacije možemo pisati:

$$F(j\omega) = R(\omega) + jX(\omega)$$

$$F(\omega) = \text{mod}\,\{F(\omega)\} = \sqrt{R^2(\omega) + X^2(\omega)}$$

$$\phi(\omega) = \text{arctg}\,\frac{X(\omega)}{R(\omega)}$$

gdje je $F(\omega)$ parna funkcija od ω a $\phi(\omega)$ je neparna funkcija od ω. Na osnovu gornjih relacija dolazimo do trećeg oblika direktne Fourier-ove transformacije:

$$F(j\omega) = F(\omega)e^{j\phi(\omega)} \tag{15.719}$$

Krive $F(\omega)$ i $\phi(\omega)$ nazivaju se zajedničkim imenom **spektar učestanosti vremenske funkcije** $f(t)$: $F(\omega)-$ *amplitudski spektar*; $\phi(\omega)-$ *fazni spektar*. Po tome se primjena Fourier-ove transformacije u elektrotehnici često naziva **spektralnom analizom**.

$$R(\omega) = F(\omega)\cos\phi(\omega) \tag{15.720}$$

$$X(\omega) = F(\omega)\sin\phi(\omega) \tag{15.721}$$

Ako relacije (15.720) i (15.721) uvrstimo u drugi oblik inverzne Fourier-ove transformacije dobijamo:

$$f(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[R(\omega)\cos\omega t-X(\omega)\sin\omega t\right]d\omega=\frac{1}{\pi}\int\limits_{0}^{\infty}\left[R(\omega)\cos\omega t-X(\omega)\sin\omega t\right]d\omega$$

$$f(t)=\frac{1}{\pi}\int\limits_{0}^{\infty}F(\omega)\left[\cos\phi(\omega)\cos\omega t-\sin\phi(\omega)\sin\omega t\right]d\omega=\frac{1}{\pi}\int\limits_{0}^{\infty}F(\omega)\cos\left[\phi(\omega)+\omega t\right]d\omega \quad (15.722)$$

Relacija (15.722) predstavlja treći oblik inverzne Fourier-ove transformacije.

Osobine (svojstva) Fourier-ove transformacije

1. **Linearnost**

   Ako je $x(t)\overset{\mathrm{F}}{\risingdotseq}X(j\omega)$ i $y(t)\overset{\mathrm{F}}{\risingdotseq}Y(j\omega)$ tada je:

$$ax(t)+by(t)\overset{\mathrm{F}}{\risingdotseq}aX(j\omega)+bY(j\omega)$$

   Ova osobina direktno slijedi iz definicije Fourier-ove transformacije.

2. **Pomak u vremenskom domenu** (TIME SHIFTING)

   Ako je $x(t)\overset{\mathrm{F}}{\risingdotseq}X(j\omega)$ tada je:

$$x(t-t_0)\overset{\mathrm{F}}{\risingdotseq}e^{-j\omega t_0}X(j\omega)$$

   Dokaz:

$$x(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

Ako t zamijenimo sa $t-t_0$ imamo:

$$x(t-t_0)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}X(j\omega)e^{j\omega(t-t_0)}d\omega=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[e^{-j\omega t_0}X(j\omega)\right]e^{j\omega t}d\omega$$

što znači da je:

$$\mathrm{F}\left\{x(t-t_0)\right\}=e^{-j\omega t_0}X(j\omega)$$

3. **Conjugation and conugate symmetry**

Ako je $x(t) \overset{\mathrm{F}}{\doteqdot} X(j\omega)$ tada je:

$$x^*(t) \overset{\mathrm{F}}{\doteqdot} X^*(-j\omega)$$

Dokaz:

$$X^*(j\omega) = \left[\int\limits_{-\infty}^{\infty} [x(t)]\, e^{-j\omega t} dt\right]^* = \int\limits_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

Zamjenjujući ω sa $-\omega$ imamo:

$$X^*(-j\omega) = \int\limits_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$$

Posledica: Ako je $x(t)$ - realan signal $(x^*(t) = x(t))$ imamo:

$$X(-j\omega) = X^*(j\omega) \tag{15.723}$$

Ova osobina se naziva *conjugate symetry*:

$$X^*(-j\omega) = \int\limits_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt = X(j\omega)$$

Zamjenjujući ω sa $-\omega$ imamo u prethodnom izrazu dobijamo relaciju 15.723.

4. **Diferenciranje u vremenskom domenu**

Ako je $x(t) \overset{\mathrm{F}}{\doteqdot} X(j\omega)$ tada je:

$$\frac{dx(t)}{dt} \overset{\mathrm{F}}{\doteqdot} j\omega X(j\omega)$$

Dokaz:

$$x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Diferenciranjem prethodnog izraza po vremenu dobijamo:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} [j\omega X(j\omega)]\, e^{j\omega t} d\omega$$

pa slijedi:

$$\frac{dx(t)}{dt} \overset{\mathrm{F}}{\doteqdot} j\omega X(j\omega)$$

Uopštenije:

$$\frac{d^n x(t)}{dt^n} \overset{\mathrm{F}}{\doteqdot} (j\omega)^n X(j\omega) \quad (n = 1, 2, 3, ...) \tag{15.724}$$

5. **Time and Frequency scaling**

Ako je $x(t) \overset{\mathrm{F}}{\risingdotseq} X(j\omega)$ tada je:

$$x(at) \overset{\mathrm{F}}{\risingdotseq} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Dokaz:

$$\mathrm{F}\left\{x(at)\right\} = \int\limits_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Uvodeći smjenu: $at = \tau$, $dt = \frac{d\tau}{a}$ dobijamo:

$$\mathrm{F}\left\{x(at)\right\} = \begin{cases} \frac{1}{a}\int\limits_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau; & a > 0 \\ -\frac{1}{a}\int\limits_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau; & a < 0 \end{cases}$$

Posledica: Ako je $a = -1$ imamo da je:

$$x(-t) \overset{\mathrm{F}}{\risingdotseq} X(-j\omega)$$

6. **Frequency shifting**

Ako je $x(t) \overset{\mathrm{F}}{\risingdotseq} X(j\omega)$ tada je:

$$e^{j\omega t_0} x(t) \overset{\mathrm{F}}{\risingdotseq} X(j\omega - j\omega_0)$$

Dokaz:

$$\mathrm{F}\left\{e^{j\omega t_0} x(t)\right\} = \int\limits_{-\infty}^{\infty} e^{j\omega t_0} x(t) e^{-j\omega t} dt = \int\limits_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt = X\left(j\omega - j\omega_0\right)$$

7. **Modulacija**

Ako je $x(t) \overset{\mathrm{F}}{\risingdotseq} X(j\omega)$ i ako su date funkcije: $g_1(t) = (\cos\omega_0 t)\, x(t)$ i $g_2(t) = (\sin\omega_0 t)\, x(t)$ tada je:

$$g_1(t) = (\cos\omega_0 t)\, x(t) \overset{\mathrm{F}}{\risingdotseq} \frac{1}{2}\left[X\left(j\omega - j\omega_0\right) + X\left(j\omega + j\omega_0\right)\right] \quad (15.725)$$

$$g_2(t) = (\sin\omega_0 t)\, x(t) \overset{\mathrm{F}}{\risingdotseq} \frac{1}{2j}\left[X\left(j\omega - j\omega_0\right) - X\left(j\omega + j\omega_0\right)\right] \quad (15.726)$$

Dokaz:

$$g_1(t) = (\cos\omega_0 t)\, x(t) = \frac{1}{2}\left[e^{j\omega_0}x(t) + e^{-j\omega_0}x(t)\right] \quad (15.727)$$

$$g_2(t) = (\sin\omega_0 t)\, x(t) = \frac{1}{2j}\left[e^{j\omega_0}x(t) - e^{-j\omega_0}x(t)\right] \quad (15.728)$$

Izrazi dati relacijama (15.725) i (15.726) slijede kao posledica primjene osobina linearnosti i pomaka u frekventnom domenu nad relacijama (15.727) i (15.728) .

8. **Osobina simetrije (symmetry)**

Ako je $x(t) \stackrel{\mathrm{F}}{\risingdotseq} X(j\omega)$ tada je:

$$x(jt) \stackrel{\mathrm{F}}{\risingdotseq} 2\pi x(-\omega)$$

Dokaz:

$$x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Zamjenom ω sa y dobijamo:

$$x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jy)e^{jyt}dy$$

Ako sada promjenljivu t zamijenimo sa promjenljivom $-\omega$ dobijamo:

$$x(-\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jy)e^{-jy\omega}dy$$

$$2\pi x(-\omega) = \int\limits_{-\infty}^{\infty} X(jy)e^{-jy\omega}dy$$

Odnosno:

$$x(jt) \stackrel{\mathrm{F}}{\risingdotseq} 2\pi x(-\omega)$$

Primjer 12:

$$e^{-\alpha|t|} \stackrel{\mathrm{F}}{\risingdotseq} \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\frac{2\alpha}{\alpha^2+t^2} \stackrel{\mathrm{F}}{\risingdotseq} 2\pi e^{-\alpha|\omega|}$$

Primjer 13:

$$e^{-at}h(t) \stackrel{\mathrm{F}}{\risingdotseq} \frac{1}{a+j\omega}, \quad a \; > \; 0$$

$$\frac{1}{a+jt} \stackrel{\mathrm{F}}{\risingdotseq} 2\pi e^{a\omega}h(-\omega), \quad a \; > \; 0$$

Primjer 14:

$$f(t) = \begin{cases} A & -\frac{b}{2} < t < \frac{b}{2} \\ 0 & |t| > \frac{b}{2} \end{cases}$$

$$F(j\omega) = \frac{2A}{\omega}\sin\frac{\omega b}{2}$$

$$\frac{2A}{t}\sin\frac{bt}{2} \stackrel{\mathrm{F}}{\risingdotseq} \begin{cases} 2\pi A, & |\omega| < \frac{b}{2} \\ 0, & |\omega| > \frac{b}{2} \end{cases}$$

8. **Izvod u frekventnom domenu**

Ako je $x(t) \stackrel{\mathrm{F}}{\risingdotseq} X(j\omega)$ tada je:

$$t^n x(t) \stackrel{\mathrm{F}}{\risingdotseq} (-1)^n \frac{d^n}{d(j\omega)^n} X(j\omega), \quad n = 1, 2, 3, ...$$

Dokaz: Diferencirajući izraz za direktnu Fourier-ovu transformaciju:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

n - puta po ω dobijamo:

$$X^{(n)}(j\omega) = \int_{-\infty}^{+\infty} (-jt)^n x(t)e^{-j\omega t}dt$$

Posmatrajući prethodni izraz slijedi da je:

$$(-jt)^n x(t) \stackrel{\mathrm{F}}{\risingdotseq} X^{(n)}(j\omega)$$

9. **Teorema konvolucije u vremenskom odmenu**

Ako je $x_1(t) \stackrel{\mathrm{F}}{\risingdotseq} X_1(j\omega)$ i $x_2(t) \stackrel{\mathrm{F}}{\risingdotseq} X_2(j\omega)$ tada je:

$$x_1(t) * x_2(t) \stackrel{\mathrm{F}}{\risingdotseq} X_1(j\omega)X_2(j\omega)$$

Dokaz:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

$$\mathrm{F}\left\{x_1(t) * x_2(t)\right\} = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right] dt$$

Uvodeći smjenu $t = \tau + y$ i zamjenjujući redosled integraljenja dobijamo:

$$\int_{-\infty}^{\infty} x_1(\tau)\int_{-\infty}^{\infty} e^{-j\omega(\tau+y)}x_2(y)dyd\tau = \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty} x_2(y)e^{-j\omega y}dy = X_1(j\omega)X_2(j\omega)$$

Posledica:

$$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)X_2(j\omega)e^{-j\omega t}d\omega$$

10. **Konvolucija u frekventnom domenu**

Ako je $x_1(t) \stackrel{\mathrm{F}}{\risingdotseq} X_1(j\omega)$ i $x_2(t) \stackrel{\mathrm{F}}{\risingdotseq} X_2(j\omega)$ tada je:

$$x_1(t) * x_2(t) \stackrel{\mathrm{F}}{\risingdotseq} \frac{1}{2\pi}\, X_1(j\omega) * X_2(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)X_2(j\omega - j\theta)d\theta$$

Dokaz:

$$\begin{aligned}
\mathrm{F}^{-1}\left\{\frac{1}{2\pi}\, X_1(j\omega) * X_2(j\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{j\omega t}\left[\int_{-\infty}^{\infty} X_1(j\theta)X_2(j\omega - j\theta)d\theta\right] d\omega = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\theta+z)t}X_2(jz)dzd\theta = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)e^{j\theta t}d\theta\ \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(jz)e^{jzt}dz = x_1(t)x_2(t)
\end{aligned}$$

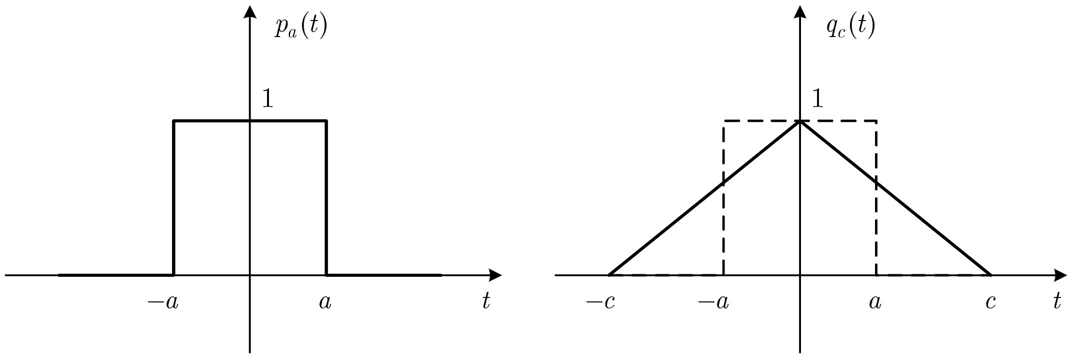
U prethodnom izrazu uvedena je smjena $\omega = \theta + z$.

Primjer 15: Signali $p_a(t)$ i $q_c(t)$ su prikazani na slici 15.329. a konvloucija signala $p_a(t)$ je jednaka:

$$p_a(t) * p_a(t) = 2aq_{2a}(t)$$

dok je signal $q_c(t)$ jednak:

$$q_c(t) = \begin{cases} 1 - \frac{|t|}{c} & |t| < c \\ 0 & |t| > c \end{cases}$$



Slika 15.329:

$$p_a(t) \stackrel{\mathrm{F}}{\risingdotseq} 2\frac{\sin(a\omega)}{\omega}$$

$$2aq_{2a}(t) \stackrel{\mathrm{F}}{\risingdotseq} \frac{4\sin^2(a\omega)}{\omega^2}$$

Ako uvedemo smjenu: $c = 2a$ dobijamo da je:

$$q_c(t) \stackrel{\mathrm{F}}{\risingdotseq} \frac{4\sin^2\left(\frac{c\omega}{2}\right)}{c\omega^2}$$

Na osnovu osobine simetrije imamo da je:

$$\frac{2\sin^2\left(\frac{ct}{2}\right)}{\pi c t^2} \stackrel{\mathrm{F}}{\risingdotseq} q_c(\omega)$$

Primjer 16: Ako je:

$$\rho(t) \stackrel{\mathrm{F}}{\risingdotseq} |F(j\omega)|^2$$

odrediti funkciju $\rho(t) =?$

$$|F(j\omega)| = F(j\omega) * F(j\omega)$$

$$\rho(t) = f(t) * f^*(-t) = \int\limits_{-\infty}^{\infty} f(t-\tau) f^*(-\tau) d\tau = \int\limits_{-\infty}^{\infty} f(t+\alpha) f^*(\alpha) d\alpha$$

Funkcija $\rho(t)$ se naziva **autokorelacija** signala $f(t)$.

11. **Prozori (Windows)**

Ako je $w(t) = 0$ za $|t| > T$ tada je:

$$F_w(j\omega) = \int\limits_{-T}^{T} f(t) w(t) e^{-j\omega t} dt$$

$$f_w(t) = f(t)w(t)$$

$$W(j\omega) = \int_{-T}^{T} w(t)e^{-j\omega t}dt$$

$$F_w(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega - jy)\, W(jy)dy$$

Primjer 17:

$$w(t) = f_T(t); \quad W(j\omega) = \frac{2\sin(T\omega)}{\omega}$$

$$F_w(j\omega) = \int_{-T}^{T} f(t)w(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} F(j\omega - jy)\frac{\sin(Ty)}{\pi y}dy$$

12. **Teorema PARSEVALA**

Ako je $x_1(t) \stackrel{\text{F}}{\risingdotseq} X_1(j\omega)$ i $x_2(t) \stackrel{\text{F}}{\risingdotseq} X_2(j\omega)$ tada je:

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)X_2^*(j\omega)d\omega$$

Dokaz: Ako u relaciju:

$$\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\omega)F_2(j\omega)e^{-j\omega t}d\omega$$

stavimo da je $t = 0$ dobijamo:

$$\int_{-\infty}^{\infty} f_1(\tau)f_2(-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\omega)F_2(j\omega)e^{-j\omega t}d\omega$$

$$\begin{aligned} x_1(t) &= f_1(t); \quad x_2^* = f_2(-t) \\ X_1(j\omega) &= F_1(j\omega); \quad X_2(j\omega) = F_2^*(j\omega) \\ & f^*(t) \stackrel{\text{F}}{\risingdotseq} F^*(-j\omega) \end{aligned}$$

12.1. **Posledica: Teorema Releja (Energy Theorem)**

Ako je $x_1(t) = x_2(t) = x(t)$ iz Parsevalove teoreme slijedi da je:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Primjer 17:

$$\frac{\sin(at)}{t} \overset{\mathrm{F}}{\doteqdot} \pi p_a(\omega)$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(at)}{t^2} dt = \frac{1}{2\pi} \int_{-a}^{a} \pi^2 d\omega = a\pi$$

Primjer 18: Ako je:

$$f(t) \overset{\mathrm{F}}{\doteqdot} F(j\omega); \quad F(j\omega) = A(\omega) e^{j\varphi(\omega)}$$

tada je:

$$\int_{-\infty}^{\infty} t^2 \ |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left[A'(\omega) \right]^2 + A^2(\omega) \left[\varphi'(\omega) \right] \right\} d\omega$$

$$\int_{-\infty}^{\infty} t \ |f(t)|^2 \, dt = -\frac{1}{2\pi} \int_{-\infty}^{\infty} A^2(\omega) \ \varphi'(\omega) d\omega$$

Fourier-ova transformacija nekih za elektrotehniku važnih vremenskih funkcija

1. **Impulsna funkcija**

Na osnovu odabiranja impulsne funkcije imamo da je:

$$\mathrm{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

2. $f(t) = A$

Na osnovu osobine simetrije iamo da je:

$$\delta(t) \overset{\mathrm{F}}{\doteqdot} 1$$
$$1 \overset{\mathrm{F}}{\doteqdot} 2\pi\delta(\omega)$$

Prema tome:

$$A \overset{\mathrm{F}}{\doteqdot} 2\pi A\delta(\omega)$$

3. $f(t) = \ \mathrm{sgn}(t)$

Ranije smo izveli da je:

$$\frac{1}{\pi t} \stackrel{\mathrm{F}}{\fallingdotseq} -j\mathrm{sgn}\omega$$

pa na osnovu osobine simetrije imamo da je:

$$\mathrm{sgn}t \stackrel{\mathrm{F}}{\fallingdotseq} \frac{2}{j\omega}$$

4. $f(t) = h(t)$

$$h(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}t$$

$$\mathrm{F}\left\{h(t)\right\} = \mathrm{F}\left\{\frac{1}{2} + \frac{1}{2}\mathrm{sgn}t\right\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$h(t) \stackrel{\mathrm{F}}{\fallingdotseq} \pi\delta(\omega) + \frac{1}{j\omega}$$

5. Fourier-ova transformacija integrala funkcije

$$\int\limits_{-\infty}^{t} f(\tau)d\tau \stackrel{\mathrm{F}}{\fallingdotseq} \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

Dokaz: Ako je $f(t) \stackrel{\mathrm{F}}{\fallingdotseq} F(j\omega)$ tada je:

$$\int\limits_{-\infty}^{t} f(\tau)d\tau = f(t) * h(t)$$

$$\begin{aligned}
\mathrm{F}\left\{\int\limits_{-\infty}^{t} f(\tau)d\tau\right\} &= \mathrm{F}\left\{f(t) * h(t)\right\} = \mathrm{F}\left\{f(t)\right\}\mathrm{F}\left\{h(t)\right\} = F(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \\
&= \pi F(j\omega)\delta(\omega) + \frac{F(j\omega)}{j\omega} = \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}
\end{aligned}$$

jer je:

$$F(j\omega)\delta(\omega) = F(0)\delta(\omega)$$

6. $f_1(t) = \cos\omega_0 t; \quad f_2(t) = \sin\omega_0 t$

$$\mathrm{F}\left\{\cos\omega_0 t\right\} = \mathrm{F}\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\} = \frac{1}{2}\mathrm{F}\left\{e^{j\omega_0 t}\right\} + \frac{1}{2}\mathrm{F}\left\{e^{-j\omega_0 t}\right\}$$

Na osnovu teorema pomaka:

$$\delta(t-a) \overset{\mathrm{F}}{\fallingdotseq} e^{-ja\omega}; \quad e^{jat} \overset{\mathrm{F}}{\fallingdotseq} 2\pi\delta(\omega - a)$$

Imamo da je:

$$\mathrm{F}\left\{\cos\omega_0 t\right\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$$\mathrm{F}\left\{\sin\omega_0 t\right\} = \mathrm{F}\left\{\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right\} = \frac{1}{2j}\mathrm{F}\left\{e^{j\omega_0 t}\right\} - \frac{1}{2j}\mathrm{F}\left\{e^{-j\omega_0 t}\right\}$$

$$\mathrm{F}\left\{\sin\omega_0 t\right\} = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

7. **Posledica teoreme konvolucije** (koristan identitet):

$$\delta(t-a) * f(t) = f(t-a)$$

8. $f_1(t) = (\cos\omega_0 t)\, h(t); \quad f_2(t) = (\sin\omega_0 t)\, h(t)$

Koristeći osobinu modulacije imamo:

$$\begin{aligned}
\mathrm{F}\left\{(\cos\omega_0 t)\, h(t)\right\} &= \frac{1}{2}\left[\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega+\omega_0) + \frac{1}{j(\omega+\omega_0)}\right] \\
&= \frac{\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + \frac{j\omega}{\omega_0^2 - \omega^2}
\end{aligned}$$

$$(\cos\omega_0 t)\, h(t) \overset{\mathrm{F}}{\fallingdotseq} \frac{\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$\begin{aligned}
\mathrm{F}\left\{(\sin\omega_0 t)\, h(t)\right\} &= \frac{1}{2j}\left[\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} - \pi\delta(\omega+\omega_0) - \frac{1}{j(\omega+\omega_0)}\right] = \\
&= \frac{\pi}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] + \frac{\omega}{\omega_0^2 - \omega^2}
\end{aligned}$$

$$(\sin\omega_0 t)\, h(t) \overset{\mathrm{F}}{\fallingdotseq} \frac{\pi}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] + \frac{\omega}{\omega_0^2 - \omega^2}$$

9. **Fourier-ova transformacija periodične funkcije**.

Ako je: $x(t)=\sum_{k=-\infty}^{\infty}\underline{a}_k e^{jk\omega_0 t}$ tada je:

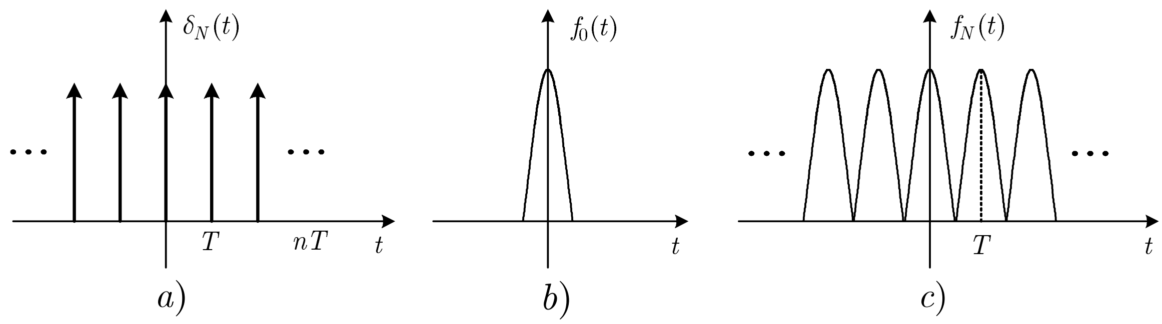
$$\mathrm{F}\left\{x(t)\right\}=X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi\underline{a}_k\delta(\omega-k\omega_0)$$

Dokaz: Ranije smo vidjeli da je:

$$e^{j\omega_0 t}\overset{\mathrm{F}}{\doteqdot}2\pi\delta(\omega-\omega_0)$$

10. **Fourier-ova transformacija povorke impulsa**.

Povorka impulsa opisana relacijom $\delta_N(t)=\sum_{n=-N}^{N}\delta\left(t+nT\right)$ prikazana je na slici 15.330a).



Slika 15.330:

$$K_n(j\omega)=\mathrm{F}\left\{\delta_N(t)\right\}=\sum_{n=-N}^{N}e^{jn\omega t}=\frac{\sin\left(N+\frac{1}{2}\right)\omega T}{\sin\left(\frac{\omega T}{2}\right)}$$

$$\omega_0=\frac{2\pi}{T}$$

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}}K_n(j\omega)d\omega=\omega_0$$

$$f_N(t)=\sum_{n=-N}^{N}f_0(t+nT)$$

Funkcije $f_0(t)$ i $f_N(t)$ su prikazane na slici 15.330b) i 15.330c) respektivno. Fourier-ova transformacija funkcije $f_N(t)$ je jednaka:

$$F_N(j\omega)=F_0(j\omega)K_N(j\omega)$$

$$\delta(t+nT) * f_0(t) = f_0(t+nT)$$

$$f_N(t) = \delta_N(t) f_0(t)$$

$$\delta_N(t) \stackrel{\mathrm{F}}{\risingdotseq} K_N(\omega) \quad N \to \infty$$

$$\overline{\delta}(t) = \sum_{n=-\infty}^{\infty} \delta(t+nT) \qquad \overline{\delta}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

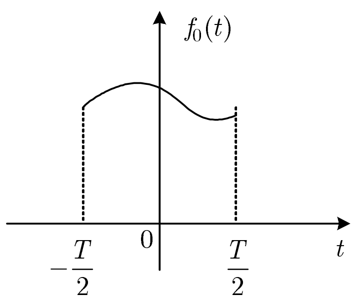
$$\overline{\delta}(t) \stackrel{\mathrm{F}}{\risingdotseq} \omega_0 \overline{\delta}(\omega)$$

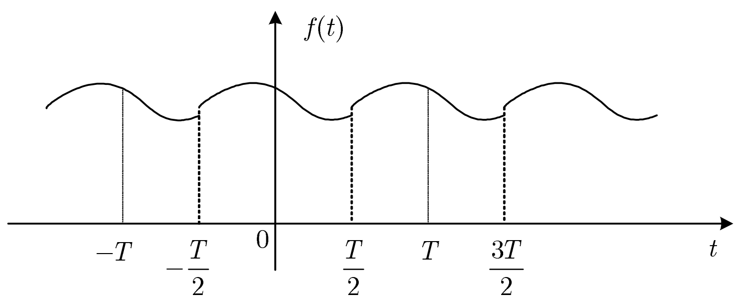
$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \stackrel{\mathrm{F}}{\risingdotseq} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}); \qquad \underline{a}_k = \frac{1}{T} \text{ za svako } k$$

Ako je funkcija $f_0(t)$ definana na sledeći način (slika 15.331):

$$f_0(t) = \begin{cases} f(t) & |t| < \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases}$$

tada je:





Slika 15.331:

$$f(t) = \sum_{n=-\infty}^{\infty} f_0(t+nT) = \overline{\delta}(t) * f_0(t)$$

$$F(j\omega) = \omega_0 \, \overline{\delta}(\omega) F_0(j\omega) = \omega_0 F_0(j\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$F(j\omega) = \omega_0 \sum_{n=-\infty}^{\infty} F_0(jn\omega_0) \delta(\omega - n\omega_0)$$

jer je:

$$F_0(j\omega)\delta(\omega - n\omega_0) = F_0(jn\omega_0)\delta(\omega - n\omega_0)$$

Veza između Fourier-ove transformacije i Fourier-ovog reda

$$\underline{a}_n = \frac{1}{T}F_0(jn\omega_0) = \frac{1}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega_0 t}dt$$

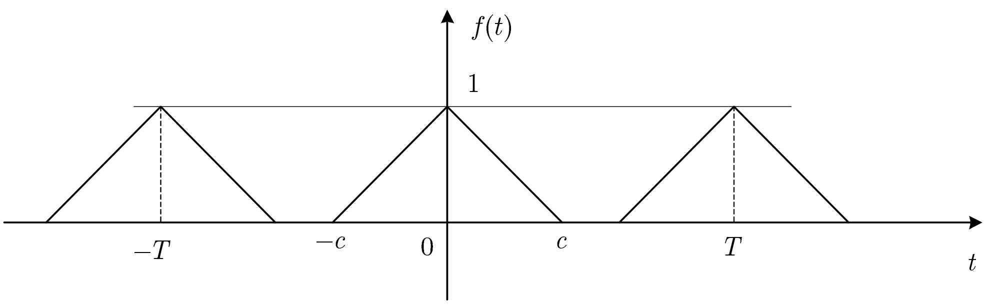
$$f(t) = \sum_{n=-\infty}^{\infty} \underline{a}_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

jer je inverzna transformacija:

$$\omega_0\delta(\omega - n\omega_0) \stackrel{\mathrm{F}^{-1}}{\risingdotseq} \frac{e^{jn\omega_0 t}}{T}$$

Primjer 19: Data je funkcija $f_0(t) = q_c(t)$.

$$F_0(j\omega) = \frac{4\sin^2\left(\frac{c\omega}{2}\right)}{c\omega^2}$$



Slika 15.332:

$$\underline{a}_n = \frac{\omega_0}{2\pi}F_0(jn\omega_0) = \frac{2\sin^2\left(\frac{n\omega_0 c}{2}\right)}{\pi\omega_0 c n^2}$$

The poisson sum formula

$$\overline{f}(t) = \sum_{n=-\infty}^{\infty} f_0(t + nT) \qquad \overline{F}(j\omega) = \sum_{n=-\infty}^{\infty} F(j\omega + jn\omega_1)$$

gdje su T i ω_1 a $\overline{F}(j\omega)$ nije Fourier-ova transformacija finkcije $\overline{f}(t)$.

$$\overline{f}(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(jn\omega_0)e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \tag{15.729}$$

Relacija (15.729) predstavlja Poisson sum formulu. Na osnovu osobine simetrije može se napisati:

$$\overline{F}(j\omega) = \frac{2\pi}{\omega_1} \sum_{n=-\infty}^{\infty} f(nT_1) e^{-jnT_1\omega}, \quad T_1 = \frac{2\pi}{\omega_1}$$

ili direktno:

$$\sum_{n=-\infty}^{\infty} \delta(\omega + n\omega_1) = \frac{1}{\omega_1} \sum_{n=-\infty}^{\infty} e^{-jnT_1\omega}$$

$$\delta(\omega + n\omega_1) * F(j\omega) = \int_{-\infty}^{\infty} \delta(\omega - y + n\omega_1) F(jy) dy = F(\omega + n\omega_1)$$

$$e^{-jnT_1\omega} * F(j\omega) = \int_{-\infty}^{\infty} e^{-jnT_1(\omega - y)} F(jy) dy = 2\pi f(nT_1) e^{-jnT_1\omega}$$

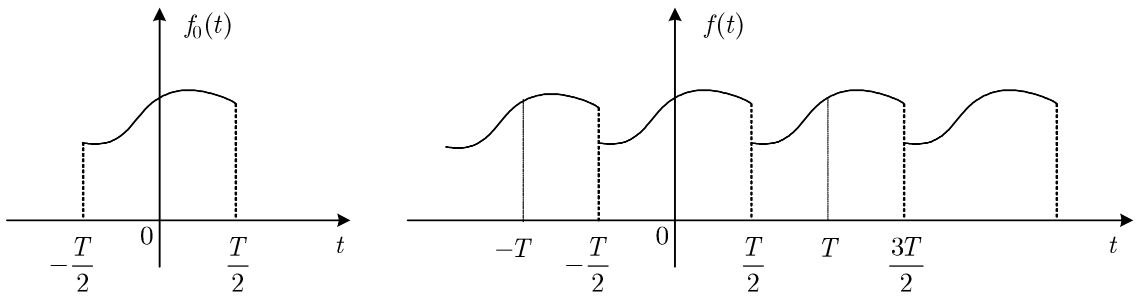
Primjer 20: Za $\omega = 0$ i $T_1 = 1$ imamo:

$$e^{-\alpha|t|} \overset{\mathrm{F}}{\risingdotseq} \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2} = \sum_{n=-\infty}^{\infty} e^{-\alpha|n|}$$

Prelaz sa Fourier-ovog reda u kompleksnom obliku na Fourier-ovu transformaciju

Neka je data neperiodična funkcija: $f(t) = f_T(t)$; $-\frac{T}{2} < t < \frac{T}{2}$; $f_T(t+T) = f_T(t)$. Funkcija $f_T(t)$ se naziva periodični razvoj funkcije $f(t)$ i obje su prikazane na slici 15.333.



Slika 15.333:

$$f_T(t) = \sum_{n=-\infty}^{\infty} \underline{c}_n e^{j\frac{2\pi nt}{T}}$$

$$\underline{c}_n = \frac{1}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f_T(x)e^{-j\frac{2\pi nx}{T}}dx$$

Ako $T \to \infty$ tada $f_T(t) \to f(t)$. Kada $T \to \infty$ tada $\omega_0 = \frac{2\pi}{T} \to 0$; $\frac{2\pi}{T} \to \Delta\omega$, tada je:

$$f_T(t) = \sum_{n=-\infty}^{\infty}\left[\frac{\Delta\omega}{2\pi}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f_T(x)e^{-j\frac{2\pi nx}{T}}dx\right]e^{jtn\Delta\omega} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f_T(x)e^{-j(x-t)n\Delta\omega}dx\right]\Delta\omega$$

$$g(\omega,t) = \frac{1}{2\pi}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f_T(x)e^{-j\omega(x-t)}dx$$

$$f(t) = \lim_{T\to\infty(\Delta\omega\to 0)}\sum_{n=-\infty}^{\infty} g(n\Delta\omega,t)\Delta\omega$$

Na osnovu fundamentalne teoreme integralnog računa:

$$f(t) = \int\limits_{-\infty}^{\infty} g(\omega,t)d\omega$$

gdje: $f_T \to f \quad T \to \infty$ imamo:

$$\lim_{T\to\infty} g(\omega,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(x)e^{-j\omega(x-t)}dx$$

$$f(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\int\limits_{-\infty}^{\infty} f(x)e^{-j\omega(x-t)}dx\right]d\omega \qquad (15.730)$$

Relacija (15.730) predstavlja Fourier-ov integral.

$$f(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\int\limits_{-\infty}^{\infty} f(x)e^{-j\omega x}dx\right]e^{j\omega t}d\omega$$

$$F(j\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad (15.731)$$

Relacija (15.731) predstavlja Direktnu Fourier-ovu transformaciju.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \tag{15.732}$$

Relacija (15.732) predstavlja Inverznu Fourier-ovu transformaciju.

Neki važni nesvojstveni integrali - integrali sa beskonačnim granicama

1. $\int\limits_0^\infty \frac{\sin \omega t}{\omega} d\omega = \begin{cases} \frac{\pi}{2} & t > 0 \\ -\frac{\pi}{2} & t < 0 \end{cases}$

2. $\int\limits_0^\infty \frac{\omega \sin \omega t}{\alpha^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-\alpha t}$

3. $\int\limits_0^\infty \frac{\cos \omega t}{\alpha^2 + \omega^2} d\omega = \frac{\pi}{2\alpha} e^{-\alpha t}$

4. $\int\limits_0^\infty \frac{\alpha \sin \omega t}{\omega(\alpha^2 + \omega^2)} d\omega = \frac{\pi}{2\alpha} \left(1 - e^{-\alpha t}\right)$

5. $\int\limits_{-\infty}^\infty \cos(\omega t)\, d\omega = 2\pi\delta(t)$

6. $\int\limits_{-\infty}^\infty \delta(\omega) e^{j\omega t} d\omega = 1; \quad \delta(t) = \frac{1}{2\pi} \int\limits_{-\infty}^\infty e^{j\omega t} d\omega$

7. $\int\limits_{-\infty}^\infty \frac{e^{jtx}}{x} dx = \begin{cases} j\pi & t > 0 \\ -j\pi & t < 0 \end{cases} \; ; \quad \int\limits_{-\infty}^\infty \frac{e^{jtx}}{x} dx = j\pi \mathrm{sgn} nt$

8. $\int\limits_0^\infty \frac{\sin x \cos ax}{x} dx = \begin{cases} \frac{\pi}{2} & |a| < 1 \\ -\frac{\pi}{4} & |a| = 1 \\ 0 & |a| > 1 \end{cases}$

9. $\int\limits_0^\infty \frac{\sin x}{\sqrt{x}} dx = \int\limits_0^\infty \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$

10. $\frac{1}{2} + \frac{1}{\pi} \int\limits_0^\infty \frac{\sin \omega t}{\omega} d\omega = h(t)$

11. $\int\limits_{-\infty}^\infty e^{-st^2} dt = \sqrt{\frac{\pi}{s}}; \qquad \mathrm{Re}\left\{\sqrt{\frac{\pi}{s}}\right\} > 0$

    $\int\limits_{-\infty}^\infty e^{j\beta t^2} dt = \sqrt{\frac{j\pi}{\beta}} = \sqrt{\frac{\pi}{\beta}} e^{j\frac{\pi}{4}}$

$\int\limits_{-\infty}^{\infty} e^{js(t+c)^2}dt = \sqrt{\frac{\pi}{s}}$; $\quad c = a + jb, \quad s = \alpha + j\beta \quad \alpha > 0$

$e^{-st^2} \overset{\mathrm{F}}{\risingdotseq} \sqrt{\frac{\pi}{s}}e^{-\frac{\omega^2}{4s}}$; $\quad \mathrm{Re}\{s\} \geq 0$. Ako je $s = \alpha$ imamo: $e^{-\alpha t^2} \overset{\mathrm{F}}{\risingdotseq} \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$

Dokaz:

$F(j\omega) = \int\limits_{-\infty}^{\infty} e^{-st^2}e^{-j\omega t}dt$

$st^2 + j\omega t = s\left(t + \frac{j\omega}{2s}\right)^2 + \frac{\omega^2}{4s}$

$F(j\omega) = e^{-\frac{\omega^2}{4s}} \int\limits_{-\infty}^{\infty} e^{-s\left(t+\frac{j\omega}{2s}\right)^2}dt = \sqrt{\frac{\pi}{s}}e^{-\frac{\omega^2}{4s}}$

12. $\int\limits_{0}^{\infty} \frac{\cos ax}{x}dx = \infty \qquad (\alpha \neq 0,\ a-$ proizvoljno$)$

13. $\int\limits_{0}^{\infty} \frac{\tan ax}{x}dx = \begin{cases} \frac{\pi}{2} & a > 0 \\ -\frac{\pi}{2} & a < 0 \end{cases}$

14. $\int\limits_{0}^{\infty} \frac{\sin^2 ax}{x}dx = \frac{\pi}{2}\,|a|$

15. $\int\limits_{-\infty}^{\infty} \sin x^2 dx = \int\limits_{0}^{\infty} \cos x^2 dx = \sqrt{\frac{\pi}{2}}$

16. $\int\limits_{0}^{\infty} \frac{\cos ax - \cos bx}{x}dx = \ln\frac{b}{a} \quad (a, b > 0)$

17. $\int\limits_{0}^{\infty} \frac{\sin x \cos ax}{x}dx = \begin{cases} \frac{\pi}{2} & |a| < 1 \\ \frac{\pi}{4} & |a| = 1 \\ 0 & |a| > 1 \end{cases}$

18. $\int\limits_{0}^{\infty} \frac{x \sin ax}{x^2+b^2}dx = \frac{\pi}{2}e^{-|ab|}\mathrm{sgn}a$

19. $\int\limits_{0}^{\infty} \frac{\cos ax}{1+x^2}dx = \frac{\pi}{2}e^{-|a|}$

Određeni integrali sa beskonačnim granicama poimaju se u smislu "uslovne vrijednosti" tj.

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \int_{-p}^{+q} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

kada $p = q \to \infty$.

LAPLACE-ova transformacija - matematički dio

Nesvojstveni integral:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt \qquad (15.733)$$

gdje je $s = \sigma + j\omega$ naziva se direktna Laplace-ova transformacija, a integral:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s) e^{st} ds \qquad (15.734)$$

inverzna Laplace-ova transformacija.

$$\begin{aligned} |f(t)| &< M e^{\sigma_0 t} \\ \operatorname{Re}\{s\} &= \sigma > \sigma_0 \end{aligned}$$

Konstante M i σ_0 su realne i pozitivne. Lapalce-ova transformacija se označava na sledeći način:

$$F(s) = \mathrm{L}\{f(t)\}$$

ili

$$f(t) \stackrel{\mathrm{L}}{\risingdotseq} F(s)$$

Inverzna Laplace-ova transformacija se označava na sledeći način:

$$f(t) = \mathrm{L}^{-1}\{F(s)\}$$

$$f(t) = \sum \operatorname*{Res}_{s_k} F(s) e^{st} \qquad (t > 0)$$

gdje su sa s_k - polovi funkcije $F(s)$.

Primjer 1: Odrediti Laplace-ovu transformaciju funkcije $f(t) = e^{-at}h(t)$.

$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}$$

$\mathrm{L}\{e^{-at}h(t)\} = \frac{1}{s+a}$ jer je $\mathrm{Re}\{s\} = \sigma > -a$ pa je integral za gornju granicu jednak nuli.

Osnovna svojstva Laplace-ove transformacije

1. **Svojstvo linearnosti**

Ako je $\mathrm{L}\{f_1(t)\} = F_1(s)$ i $\mathrm{L}\{f_2(t)\} = F_2(s)$ onda je:

$$\mathrm{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

$$c_1 f_1(t) + c_2 f_2(t) = c_1 \mathrm{L}^{-1}\{F_1(s)\} + c_2 \mathrm{L}^{-1}\{F_2(s)\}$$

2. **Teorema pomaka**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$

3. **Teorema kašnjenja**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\{f(t-\tau)h(t-\tau)\} = e^{-s\tau}F(s)$$

4. **Teorema skaliranja**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right) \qquad c > 0$$

5. **Diferenciranje u vremenskom domenu**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$\mathrm{L}\left\{f^{(n)}(t)\right\} = \mathrm{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(t) s^{(n-k)}$$

6. **Integracija u vremenskom domenu**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

$$\mathrm{L}\left\{\int_0^t \cdots \int_0^t f(t)\,(dt)^n\right\} = \frac{F(s)}{s^n}$$

7. **Integracija u kompleksnom domenu**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)ds$$

8. **Diferenciranje po parametru**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{\frac{\partial}{\partial x} f(t,x)\right\} = \frac{\partial}{\partial x} F(s,x)$$

9. **Integracija po parametru**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{\int_{x_0}^{x} f(t,x)dx\right\} = \int_{x_0}^{x} F(s,x)dx$$

10. **Konvolucija**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ i $\mathrm{L}\{g(t)\} = G(s)$ tada je:

$$\mathrm{L}\{f(t) * g(t)\} = F(s)G(s)$$

$$\mathrm{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s)$$

$$\mathrm{L}^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(\tau)g(t-\tau)d\tau$$

Dokaz:

$$F(s) = \int_0^\infty f(\tau)e^{-st}d\tau$$

$$F(s)G(s) = \int_0^\infty f(\tau)\left[G(s)e^{-st}\right]d\tau$$

$$F(s)G(s) = \int_0^\infty f(\tau)\mathrm{L}\left\{g(t-\tau)h(t-\tau)e^{-st}dt\right\}d\tau = \int_0^\infty f(\tau)\left[\int_0^\infty g(t-\tau)h(t-\tau)e^{-st}dt\right]d\tau$$

Kako je $h(t-\tau)=0$ za $\tau > t$ dobijamo:

$$F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt$$

iz čega slijedi da je:

$$F(s)G(s) = \mathrm{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]$$

11. **Diferenciranje u kompleksnom domenu**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ tada je:

$$\mathrm{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^nF(s)}{ds^n}; \qquad n = 0, 1, 2, 3, ...$$

12. **Laplace-ova transformacija periodične funkcije**

Ako je $\mathrm{L}\{f(t)\} = F(s)$ i $f(t+T) = f(t)$ tada je:

$$\mathrm{L}\left\{f(t)\right\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$$

Ako je $f(t) = p(t)$, $0 < t < T$, $p(t) = f(t)\left[h(t) - h(t-T)\right]$ i $f(t+T) = f(t)$ imamo:

$$\mathrm{L}\left\{f(t)\right\} = \frac{\mathrm{L}\left\{p(t)\right\}}{1-e^{-sT}}$$

13. **Teoreme o graničnim vrijednostima**

Ako je funkcija $f(t)$ neprekidna i kada u tački $t = 0$ ima konačni diskontinuitet onda važi:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Ako egistira $\lim_{t\to\infty} f(t) < \infty$ onda važi:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Dokazi:

Dokaz za prvu teoremu:

$$\mathrm{L}\left\{f'(t)\right\} = \int_{0^-}^{\infty} f'(t)e^{-st}dt = sF(s) - f(0^-)$$

Ako je $f(t)$ neprekidna u $t = 0$ odnosno $f(0^+) = f(0^-)$ onda imamo:

$$\lim_{s\to\infty}\int_{0^-}^{\infty} f'(t)e^{-st}dt = \int_{0^-}^{\infty} f'(t)\left[\lim_{s\to\infty} e^{-st}\right]dt = 0 = \lim_{s\to\infty} sF(s) - f(0^-)$$

$$\lim_{s\to\infty} sF(s) = f(0^-) = f(0^+)$$

Ako $f(t)$ ima konačni diskontinuitet u $t = 0$ onda možemo pisati: $f(t) = g(t) + Ah(t)$ gdje za funkciju $g(t)$ važi $g(0^+) = g(0^-)$ i za $t < 0$ $f(t) = g(t)$. Onda je $f(0^-) = g(0^-)$ i $A(0^+) = g(0^+) + A$, odnosno:

$$A = f(0^+) - g(0^+) = f(0^+) - g(0^-) = f(0^+) - f(0^-)$$

$$sF(s) = sG(s) + A$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} sG(s) + A = g(0^+) + A = f(0^+)$$

Dokaz za drugu teoremu:

$$\lim_{s\to 0} sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t)dt = \lim_{t\to\infty} f(t) - f(0^-)$$

iz čega slijedi da je:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Na sledećim primjerima ćemo ispitati da li važe ili ne teoreme o graničnim vrijednostima.

Primjer 2:

$$F(s) = \frac{4(s+1)}{s^2+2s+5}$$

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{4s(s+1)}{s^2+2s+1} = 4$$

$$f(t) = \mathrm{L}^{-1}\left\{F(s)\right\} = 4e^{-t}\cos 2t$$

$$f(0^+) = 4$$

Primjer 3:

$$f(t) = \delta(t) + 4e^{-t}$$

$$F(s) = 1 + \frac{4}{s+1} = \frac{s+5}{s+4}$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s(s+5)}{s+4} = \infty$$

$$f(0^+) = 4$$

Teorema **ne važi** jer funkcija $f(t)$ u tački $t = 0$ nema konačni diskontinuitet već beskonačni.

Primjer 4:

$$F(s) = \frac{5s+2}{s(s+1)}$$

$$\lim_{s\to 0} sF(s) = 2$$

$$f(t) = 2h(t) + 3e^{-t}$$

$$\lim_{t\to\infty} f(t) = 2$$

Teorema važi.

Primjer 5:

$$F(s) = \frac{1}{s-1}$$

$$\lim_{s\to 0} sF(s) = 0$$

$$f(t) = e^t$$

$$\lim_{t\to\infty} f(t) = \infty$$

Teorema ne važi.

14. **Konvolucija u kompleksnom domenu**

$$\mathrm{L}\left\{f(t)g(t)\right\} = \frac{1}{2\pi j} F(s) * G(s)$$

$$F(s) * G(s) = \int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(z)G(s-z)dz = \int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(s-z)G(z)dz$$

Dokaz: Po definiciji je:

$$f(t)g(t) = \left[\frac{1}{2\pi j}\int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(z)e^{zt}dz\right] g(t) = \frac{1}{2\pi j}\int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(z) \underbrace{e^{zt}g(t)}_{\mathrm{L}^{-1}\{G(s-z)\}} dz$$

$$f(t)g(t) = \frac{1}{2\pi j}\int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(z)\left[\mathrm{L}^{-1}\left\{G(s-z)\right\}\right] dz = \mathrm{L}^{-1}\left\{\frac{1}{2\pi j}\int\limits_{\sigma - j\omega}^{\sigma + j\omega} F(z)G(s-z)dz\right\}$$

$$f(t)g(t) = \mathrm{L}^{-1}\left\{\frac{1}{2\pi j} F(s) * G(s)\right\}$$

15. **Inverzna Laplace-ova transformacija racionalnih funkcija**

Hevisajdov razvoj: Ako je data funkcija $F(s)$ oblika:

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

i ako je $m < n$, a s_k su prosti korjeni polinoma $B(s) = 0$ onda je funkcija $f(t)$:

$$f(t) = \mathrm{L}^{-1}\left\{F(s)\right\} = \sum_{k=1}^{n} \frac{A(s_k)}{B(s_k)} e^{s_k t}$$

za $t > 0$. Ako $B(s) = 0$ ima korjen $s = 0$ tj. $B(s) = sB_1(s)$ onda je funkcija $f(t)$ jednaka:

$$f(t) = \mathrm{L}^{-1}\left\{F(s)\right\} = \frac{A(0)}{B_1(0)} + \sum_{k=2}^{n} \frac{A(s_k)}{B(s_k)} e^{s_k t}$$

za $t > 0$. Ako $B(s) = 0$ ima višestruke korjene tj: $B(s) = b_n(s - s_1)^{k_1}(s - s_2)^{k_2} \cdots (s - s_l)^{k_l}$ gdje je $k_1 + k_2 + \cdots k_l = n$ tada je:

$$f(t) = \mathrm{L}^{-1}\{F(s)\} = \sum_{i=1}^{l} \frac{1}{(k_l - 1)!} \lim_{s \to s_i} \frac{d^{k_i - 1}}{ds^{k_i - 1}} \left[(s - s_i)^{k_i} F(s) e^{st}\right]$$

16. **Rezidijum**

$$f(t) = \sum \underset{s_k}{\mathrm{Res}} F(s) e^{st} \qquad t > 0$$

Ako se funkcija $F(s)$ može zapisati u obliku pravog razlomka tj:

$$F(s) = \frac{F_1(s)}{F_2(s)}$$

i ako je pol s_k reda m imamo:

$$\underset{s_k}{\mathrm{Res}} \frac{F_1(s)}{F_2(s)} e^{st} = \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{ds^{m-1}} \frac{F_1(s)}{F_2(s)} (s - s_k)^m e^{st} \right]_{s = s_k}$$

$$\underset{s_k}{\mathrm{Res}} \frac{F_1(s)}{F_2(s)} e^{st} = e^{s_k t} \sum_{i=1}^{m} \frac{t^{(m-i)} A^{(i-1)}(s_k)}{(m-i)!(i-1)!}$$

Posledica: Ako funkcija $F(s)$ u imeniocu ima faktor $(s + a)^n$ onda je:

$$F(s) = \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \cdots + \frac{A_1}{s+a} + F_1(s)$$

$$A_k = \frac{1}{(n-k)!} \frac{d^{n-k}}{ds^{n-k}} \left[(s+a)^n F(s)\right]_{s=-a}$$

Ako funkcija $F(s)$ sadrži kompleksne korjene oni se uvijek javljaju u kompleksnim parovima $s = \alpha \pm j\beta$:

$$F(s) = \frac{A}{s - \alpha - j\beta} + \frac{B}{s - \alpha + j\beta}$$

$$A = (s - \alpha - j\beta) F(s)\big|_{s = \alpha + j\beta}$$

$$B = (s - \alpha + j\beta) F(s)\big|_{s = \alpha - j\beta}$$

$$B = \underline{A}^*$$

$$f(t) = A e^{(\alpha + j\beta)t} + A^* e^{(\alpha - j\beta)t}$$

$$A = |A|\, e^{j\theta}$$

$$f(t) = 2\,\mathrm{Re}\left\{Ae^{(\alpha + j\beta)t}\right\} = 2\,\mathrm{Re}\left\{|A|\, e^{\alpha t} e^{j(\beta t + \theta)}\right\} = 2\,|A|\, e^{\alpha t}\cos(\beta t + \theta)$$

Drugi način nalaženja invrerzne Laplace-ove transformacije (originala) pri postojanju višestrukih polova s_k koji ne zahtijeva diferenciranje, sastoji se u razvoju racionalno-razlomljene funkcije na djelimične (parcijalne) proste razlomke po poznatim metodama. Za prelaz od prostih razlomaka ka originalu koriste se relacije:

$$\frac{1}{(s - s_k)^i} \stackrel{\mathrm{L}}{\doteqdot} \frac{1}{(i-1)!} t^{i-1} e^{s_k t}$$

Za razvoj $\frac{F_1(s)}{F_2(s)}$ na djelimične razlomke pri postojanju $m-$ to strukog pola s_k može se koristiti formula:

$$\frac{F_1(s)}{F_2(s)} = \sum_{i=1}^{m} \frac{K_i}{(s - s_k)^i}$$

$$K_i = \frac{1}{(m-1)!} A^{(m-i)}(s_k)$$

što dovodi do ranije naveden formule.

Primjer 6: Primjenom inverzne Laplace-ove transformacije odrediti funkciju $f(t)$ ako je $F(s)$ jednako:

$$F(s) = \frac{6}{s^4(s+1)}$$

Znamo da je:

$$\mathrm{L}\{t^n\} = \frac{1}{s^n}\mathrm{L}\{n!\} = \frac{n!}{s^{n+1}}; \quad n = 1, 2, 3, ...$$

Razvojem funkcije $F(s)$ na parcijalne razlomke dobijamo:

$$F(s) = \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1}$$

ili putem "LONG DIVISION" za realne polove višeg reda:

$$F(s) = \frac{6}{s^4}\left[\frac{1}{s+1}\right] = \frac{6}{s^4}\left[1 - s + s^2 - s^3 + \frac{s^4}{s+1}\right] = \frac{6}{s^4} - \frac{6}{s^3} + \frac{6}{s^2} - \frac{6}{s} + \frac{6}{s+1}$$

$$F(s) = \frac{3!}{s^4} - 3\frac{2!}{s^3} + 6\frac{1!}{s^2} - \frac{6}{s} + \frac{6}{s+1}$$

$$f(t) = \mathrm{L}^{-1}\{F(s)\} = t^3 - 3t^2 + 6t - 6 + 6e^{-t}$$

Primjer 7:

$$\frac{F(s)}{G(s)} = \frac{\omega^3}{(s^2+\omega^2)^2} = \frac{\omega^3}{(s+j\omega)^2(s-j\omega)^2}$$

Polovi su: $s_1 = j\omega$ i $s_2 = -j\omega$. Za nalaženje inverzne Laplace-ove transformacije funkcije koja ima višestruke polove ($p > 1$) koristimo:

$$\mathrm{L}^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{l=1}^{l=p} \underline{K}_l \frac{t^{p-l}}{(p-l)!} e^{\underline{s}_1 t} + \sum_{l=1}^{l=p} \underline{K}'_l \frac{t^{p-l}}{(p-l)!} e^{\underline{s}_2 t}$$

$$\underline{K}_l = \lim_{s \to s_1} \frac{1}{(l-1)!} \frac{d^{l-1}}{ds^{l-1}} \left[(s-s_1)^p \frac{F(s)}{G(s)}\right] \qquad l = 1, 2, ..., p$$

$$\underline{K}'_l = \lim_{s \to s_2} \frac{1}{(l-1)!} \frac{d^{l-1}}{ds^{l-1}} \left[(s-s_2)^p \frac{F(s)}{G(s)}\right] \qquad l = 1, 2, ..., p$$

$$\underline{K}_1 = -\frac{\omega}{4}; \quad \underline{K}_2 = \frac{1}{j4}$$

$$\underline{K}'_1 = -\frac{\omega}{4}; \quad \underline{K}'_2 = -\frac{1}{j4}$$

$$\mathrm{L}^{-1}\left\{\frac{F(s)}{G(s)}\right\} = -\frac{\omega}{4} t e^{j\omega t} + \frac{1}{j4} e^{j\omega t} - \frac{\omega}{4} t e^{-j\omega t} - \frac{1}{j4} e^{-j\omega t} = -\frac{\omega t}{2} \cos\omega t + \frac{1}{2} \sin\omega t$$

VAŽNA NAPOMENA: Ako egzistira Fourier-ova transformacija funkcije $f(t) \overset{\mathrm{F}}{\fallingdotseq} F(j\omega)$ onda sigurno egistira i Laplace-ova transformacija $f(t) \overset{\mathrm{L}}{\fallingdotseq} F(s)$ jer su uslovi egzitencije Fourier-ove transformacije rigorozniji od uslova za egzitenciju Laplace-ove transformacije. Obrnuto ne važi. Laplace-ova transformacija je u velikoj upotrebi za analizu električnih kola. Postoje bogato urađene tablice za parove Laplace-ovih transformacija. Mnogi autori kažu da se one mogu koristiti i za određivanje Fourier-ove transformacije zamjenjujući $s \to j\omega$. Na primjer:

$$\mathrm{L}\{h(t)\} = \frac{1}{s}$$

$$\mathrm{F}\{h(t)\} = \frac{1}{j\omega}$$

to je pogrešno jer znamo da je Fourier-ova transformacija funkcije $h(t)$ jednaka:

$$\mathrm{F}\{h(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

Pravilno je:

$$\mathrm{F}\left\{h(t)\right\} = F(j\omega) = \mathrm{L}\left\{h(t)\right\}_{s=j\omega} + \frac{1}{2}\sum_{k} \operatorname*{Res}_{s_k}\left\{F(s)\right\}\ 2\pi\delta(\omega-\omega_k)$$